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Breaking of relativistic simple waves

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The breaking of relativistic simple waves, for one-dimensional flow in the space–time of special relativity, is investigated. The cases of relativistic acoustic and magnetoacoustic waves are treated in detail and the critical time for breaking is evaluated.

1. Introduction

Simple waves are the nonlinear analogue of the plane travelling waves of linear theory. Under certain conditions, the profile of a simple wave steepens, owing to the nonlinearity of the evolution equations, and the wave breaks. Under suitable conditions this phenomenon can be interpreted as implying shock formation (Whitham 1974). Simple waves are conceptually very important for a nonlinear theory: they represent exact solutions of the nonlinear evolution equations and, for hyperbolic systems, their behaviour provides essential clues for investigating the properties of the general wave solutions (as asymptotic waves) (Majda & Rosales 1984). Also, by using simple waves and suitable discontinuities it is possible to construct solutions for the Riemann (shock-tube) problem (Thompson 1986).

Relativistic fluid dynamics and magneto-fluid dynamics (MFD) are fields of increasing importance for several applications in astrophysics, as, for example, in the models of extragalactic radio sources (Begelman, Blandford & Rees 1984; Ferrari, Trussoni & Zaninetti 1983), and in gravitational collapse (Hawley, Smarr & Wilson 1984), in plasma physics, as in the case of strong ionizing shocks (Taussig 1973) and intense charged particle beams (Miller 1982), in nuclear physics, as in the case of heavy ion reactions (Amsden *et al.* 1978) and phase transitions to a quark–gluon plasma (Clare & Strottman 1986).

In this paper we shall investigate the breaking of a relativistic simple wave, which is essential for understanding the formation of discontinuities in relativistic fluid dynamics and magneto-fluid dynamics.

This problem is obviously relevant for astrophysics. In particular, the solutions which we shall investigate could be used as benchmarks for testing general relativistic MFD numerical codes used in astrophysics (Hawley *et al.* 1984), in the same way as the solutions of the relativistic shock-tube problem are used for general relativistic fluid dynamics codes. We notice that the results presented in this paper could be of interest also for laboratory plasma physics, in the area of plasma heating by strong shocks.

The plan of the paper is the following: in §2 we recall some basic results from the theory of simple waves; in §3 we compute the critical time for classical fluid dynamics and MFD. These results will be used for comparison with the relativistic ones.

In §4 we investigate the breaking of simple waves in relativistic fluid dynamics and MFD.

Finally in §5 we describe our numerical computations, and conclusions are drawn.

2. Simple waves and breaking time

Let us consider the following quasilinear system of partial differential equations in $\mathbb{R}^4$ (with coordinates x^α, where $x^0 = t$ can be interpreted as time and x^i are spatial coordinates in a given inertial frame):

$$\boldsymbol{A}^\alpha \partial_\alpha \boldsymbol{u} = 0, \tag{2.1}$$

where $\boldsymbol{u} = (u^1, \ldots, u^N)^{\mathrm{T}}$ is the field vector and $\boldsymbol{A}^\alpha = \boldsymbol{A}^\alpha(\boldsymbol{u})$ are $N \times N$ smooth matrices, the superscript T denotes transposition and ∂_α are the partial derivatives with respect to x^α. In the following, greek indices will run from 0 to 3 and latin ones from 1 to 3.

A simple wave solution is a smooth solution (Jeffrey 1976; Boillat 1970)

$$\boldsymbol{u} = \boldsymbol{u}(\phi), \tag{2.2}$$

with

$$\phi = \phi(x^\alpha). \tag{2.3}$$

Then (2.1) yields

$$\boldsymbol{A}^\alpha \partial_\alpha \phi \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}\phi} = 0. \tag{2.4}$$

We introduce the normal speed of propagation λ and the normal n^i through

$$\lambda = -\frac{\partial_t \phi}{|\boldsymbol{\nabla}\phi|}, \quad n^i = \frac{\partial_i \phi}{|\boldsymbol{\nabla}\phi|}, \tag{2.5}$$

where $|\boldsymbol{\nabla}\phi| = (\partial_i \phi\, \partial_j \phi\, \delta^{ij})^{\frac{1}{2}}$.

Then (2.4) can be rewritten as

$$(\boldsymbol{A}^i n_i - \lambda \boldsymbol{A}^0) \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}\phi} = 0.$$

Let $\lambda^{(k)}$ be a simple eigenvalue of the matrix $\boldsymbol{A}_n = \boldsymbol{A}^i n_i$ and $\boldsymbol{d}^{(k)}$ the corresponding eigenvector with respect to the matrix $\boldsymbol{A}^0$ (assumed non-singular), i.e. $\lambda^{(k)}$ *and* $\boldsymbol{d}^{(k)}$ satisfy

$$(\boldsymbol{A}^i n_i - \lambda^{(k)} \boldsymbol{A}^0)\, \boldsymbol{d}^{(k)} = 0. \tag{2.6}$$

Equation (2.4) will be satisfied by

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}\phi} = \pi(\phi)\, \boldsymbol{d}^{(k)}(\boldsymbol{u}, n^i), \tag{2.7}$$

where $\pi(\phi)$ is a proportionality factor.

Notice also that, from (2.5), there follows for the phase ϕ corresponding to $\lambda^{(k)}$,

$$\partial_t \phi + |\boldsymbol{\nabla}\phi|\, \lambda^{(k)}(\boldsymbol{u}, n^i) = 0. \tag{2.8}$$

If one were able to determine explicitly $N-1$ first integrals of (2.7) $J_1(\boldsymbol{u}) =$ constant, $\ldots, J_{N-1}(\boldsymbol{u}) =$ constant, one would have found an explicit simple wave relationship (2.2); then (2.8) would give an explicit expression for $\phi = \phi(x^\alpha)$.

The functions $J_1(\boldsymbol{u}), \ldots, J_{N-1}(\boldsymbol{u})$ reduce to the well-known Riemann invariants when $N = 2$ (e.g. in the case of isentropic one-dimensional gas dynamics). In general these first integrals are called generalized Riemann invariants (Jeffrey 1976) or simply Riemann invariants. Notice that $J_1, \dots, J_{N-1}$ are constant along the $\lambda^{(k)}$-characteristics.

In the following we shall consider one-dimensional propagation, $\boldsymbol{n} = \text{constant}$; therefore $\lambda^{(k)}$ is independent of n^i. It is easy to show that it is not restrictive to take ϕ of the form

$$\phi(x,t) = x - \lambda(\boldsymbol{u})\,t, \tag{2.9}$$

taking

$$n^i = (1,0,0). \tag{2.10}$$

Notice that λ, in general, is not a constant and the initial profile changes its shape while it propagates.

The goal of this paper is to calculate the earliest time at which the profile breaks (becomes multivalued). In order to achieve this, let us rewrite (2.8) in characteristic form (Whitham 1974, p. 20):

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = 0 \quad \text{on} \ \frac{\mathrm{d}x}{\mathrm{d}\tau} = \lambda^{(k)}[\boldsymbol{u}(\phi)]. \tag{2.11}$$

Along each characteristic the field $\boldsymbol{u}$ is constant, as follows easily from

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}\tau} = \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}\phi}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = 0.$$

Then the characteristics are straight lines in the plane (x,t), with slope $\lambda^{(k)}[\boldsymbol{u}(\phi)]$ and equation

$$x = \xi + \lambda^{(k)}[\boldsymbol{u}(\phi)]\,t, \tag{2.12}$$

with

$$x(t=0) = \xi.$$

Let us consider for (2.8) the initial-value problem

$$\phi(0,x) = f(x) \quad (x \in \mathbb{R}).$$

If a characteristic intersects the x-axis at $x = \bar{\xi}$, then

$$\phi = f(\bar{\xi})$$

along the whole characteristic according to (2.11).

Allowing $\bar{\xi}$ to vary, we obtain

$$\left.\begin{aligned} &\phi = f(\bar{\xi}), \\ &\bar{\xi} = x - \lambda^{(k)}[\boldsymbol{u}(f(\bar{\xi}))]\,t, \end{aligned}\right\} \tag{2.13}$$

which is the solution in implicit form.

A break occurs when two characteristics intersect each other: this implies (Whitham 1974, p. 23) the condition

$$1 + \frac{\mathrm{d}\lambda^{(k)}}{\mathrm{d}\xi}t = 0. \tag{2.14}$$

The infimum positive value of t is called the breaking time (t_{B})

$$\left.\begin{aligned} t_{\mathrm{B}} &= \inf\left[-\frac{1}{\mathrm{d}\lambda^{(k)}/\mathrm{d}\xi}\right], \\ \lambda^{(k)} &= \lambda^{(k)}[\boldsymbol{u}(f(\xi))]. \end{aligned}\right\} \tag{2.15}$$

In the following we shall give as initial condition

$$f(\xi) = a \sin\frac{2\pi}{d}\xi \quad (\xi \in [0, d]), \tag{2.16}$$

with a the amplitude and d the wavelength.

3. Classical magneto-fluid dynamics: one-dimensional flow

Let us consider a compressible fluid in the presence of a magnetic field $\boldsymbol{H}$.

When viscosity, thermal conductivity and electrical resistivity are neglected, the motion is described by the equations (Cabannes 1970)

$$\left.\begin{aligned} \rho\left[\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v}\right] &= -\nabla p + (\nabla\times\boldsymbol{H})\times\mu\boldsymbol{H}, \\ \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{v}) &= 0, \\ \frac{\partial s}{\partial t} + \boldsymbol{v}\cdot\nabla s &= 0, \\ \frac{\partial\boldsymbol{H}}{\partial t} - \nabla\times(\boldsymbol{v}\times\boldsymbol{H}) &= 0, \end{aligned}\right\} \tag{3.1}$$

where $\boldsymbol{v}$ is the fluid velocity, ρ the mass density, s the entropy, μ is the magnetic permeability (assumed constant) and p is the pressure, which is an assigned function of ρ and s.

For one-dimensional flow, the system (3.1) can be written in the matrix form (2.1):

$$\boldsymbol{A}^0\partial_t\boldsymbol{u} + \boldsymbol{A}^1\partial_x\boldsymbol{u} = 0,$$

where

$$\boldsymbol{A}^0 = \boldsymbol{I} \text{ (identity matrix)},$$

$$\boldsymbol{u} = (H_x, H_y, H_z, v_x, v_y, v_z, \rho, s)^{\mathrm{T}},$$

$$\boldsymbol{A}^1 = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -v_y & v_x & 0 & H_y & -H_x & 0 & 0 & 0 \\ -v_z & 0 & v_x & H_z & 0 & -H_x & 0 & 0 \\ 0 & \dfrac{\mu H_y}{\rho} & \dfrac{\mu H_z}{\rho} & v_x & 0 & 0 & \dfrac{\partial p}{\rho\,\partial\rho} & \dfrac{\partial p}{\rho\,\partial s} \\ 0 & -\dfrac{\mu H_x}{\rho} & 0 & 0 & v_x & 0 & 0 & 0 \\ 0 & 0 & -\dfrac{\mu H_x}{\rho} & 0 & 0 & v_x & 0 & 0 \\ 0 & 0 & 0 & \rho & 0 & 0 & v_x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_x \end{vmatrix}. \tag{3.2}$$

We observe that the component H_x of the vector $\boldsymbol{u}$ is not an unknown of the problem, because we have

$$\frac{\partial H_x}{\partial t} = 0 \Rightarrow H_x = f(x).$$

According to (2.6), (2.10) the eigenvalues of the matrix $\boldsymbol{A}_n = \boldsymbol{A}^1$ are then

$$\lambda^{(0)} = v_x \text{ (material waves)}, \quad \lambda_{\mathrm{A}}^{(\pm)} = v_x \pm \left(\frac{\mu H_x^2}{\rho}\right)^{\frac{1}{2}} \text{ (Alfvén waves)},$$

$$\lambda_{\mathrm{M}}^{(\pm)} = v_x \pm \left(\frac{1}{2}\left[c_s^2 + \frac{\mu H^2}{\rho} \pm \left\{\left(c_s^2 + \frac{\mu H^2}{\rho}\right)^2 - 4c_s^2 \frac{\mu H_x^2}{\rho}\right\}^{\frac{1}{2}}\right]\right)^{\frac{1}{2}}, \tag{3.3}$$

(magnetoacoustic waves, fast and slow)

with $c_s^2 = (\partial p/\partial \rho)|_s$ the sound speed.

The state equation coupled to (3.1) will be the polytropic one:

$$p = K\rho^{\gamma}, \tag{3.4}$$

where γ is the polytropic index, $K = \exp(s/c_v)$, where c_v is the specific heat at constant volume. In this case the sound velocity is

$$c_s^2 = K\gamma\rho^{\gamma-1}.$$

In the following we shall consider simple waves, and the associated breaking time, in some particular cases.

3.1. *Fluid-dynamic case*

For $\boldsymbol{H} \equiv 0$ we obtain the fluid-dynamic case, and for purely longitudinal motion $\boldsymbol{u} = (\rho, s, v_x)^{\mathrm{T}}$. The eigenvalues (3.3) reduce to

$$\lambda^{(0)} = v_x \text{ (material waves)}, \quad \lambda^{(\pm)} = v_x \pm c_s \text{ (acoustic waves)}.$$

The Riemann invariants associated with acoustic waves are

$$J_1 = s, \quad J_{\pm} = v_x \mp \int \frac{c_s}{\rho} \mathrm{d}\rho, \tag{3.5a, b}$$

with the $\mp$ sign referring to $\lambda^{(\pm)}$.

From (3.5*b*) using (3.4), we can obtain the useful relationships (Landau & Lifshitz 1959):

$$c_s = c_{s0} \pm \tfrac{1}{2}(\gamma - 1) v_x, \tag{3.6}$$

$$\rho = \rho_0 \left[1 \pm \tfrac{1}{2}(\gamma-1)\frac{v_x}{c_{s0}}\right]^{\frac{2}{\gamma-1}}. \tag{3.7}$$

c_{s0} and ρ_0 are respectively the sound velocity and the density, calculated at $v_x = 0$. From (3.7) we know ρ in terms of v_x, and because $s = J_1$ we obtain the simple wave relationship (2.2):

$$\boldsymbol{u} = \boldsymbol{u}(v_x),$$

and in this case

$$\phi = v_x.$$

Formula (3.7) gives us also a constraint on v_x:

$$1 \pm \tfrac{1}{2}(\gamma-1)\frac{v_x}{c_{s0}} \geqslant 0, \tag{3.8}$$

otherwise cavities and instabilities may occur in the fluid.

Now let us calculate the breaking time (2.15) for the simple wave solution (3.7):

$$\frac{\mathrm{d}\lambda^{(\pm)}}{\mathrm{d}\xi} = \frac{\mathrm{d}\lambda^{(\pm)}}{\mathrm{d}v_x}\frac{\mathrm{d}v_x}{\mathrm{d}\xi} = \tfrac{1}{2}(\gamma+1)\frac{\mathrm{d}v_x}{\mathrm{d}\xi},$$

but according to (2.16)

$$\phi = v_x = f(\xi) = v_0 \sin\frac{2\pi}{d}\xi \quad (\xi \in [0, d]).$$

Then we have

$$t_B = \inf_{\xi} -\frac{d}{(\gamma+1)\, v_0 \pi \cos\frac{2\pi}{d}\xi} = \frac{d}{(\gamma+1)\pi v_0}. \tag{3.9}$$

Hence t_B is a decreasing function of the amplitude v_0 and increases with the wavelength d.

3.2. *MFD case with* $v_y = v_z = 0, H_x = 0$

For the sake of definiteness we have chosen the case in which the motion is purely longitudinal and associated only with magnetoacoustic waves, with $\boldsymbol{u} = (H_y, H_z, v_x, \rho, p)^T$.

Then from (3.3) the slow magnetoacoustic waves coincide with the material wave, and for the fast waves one has

$$\lambda_M^{(\pm)} = v_x \pm \left(c_s^2 + \frac{\mu H^2}{\rho}\right)^{\frac{1}{2}}. \tag{3.10}$$

The associated Riemann invariants are

$$J_1 = s, \quad J_2 = \frac{H_y}{\rho}, \quad J_3 = \frac{H_z}{\rho}, \quad J_{\pm} = v_x \mp \int \left(c_s^2 + \frac{\mu H^2}{\rho}\right)^{\frac{1}{2}} \mathrm{d}\rho, \tag{3.11a–d}$$

(with the sign $\mp$ associated to $\lambda_M^{(\pm)}$).

It is convenient to write (3.10), (3.11) in a dimensionless form by introducing the density at the stagnation point, ρ_0. Then we can define

$$\bar{\rho} = \frac{\rho}{\rho_0}, \quad \bar{v}_x = \frac{v_x}{c_{s0}}, \quad \mathscr{W} = \frac{\mu H_0^2/\rho_0}{c_{s0}^2} = \frac{\mu(J_2^2 + J_3^2)\,\rho_0}{c_{s0}^2},$$

where the suffix 0 indicates that the quantities are evaluated at the stagnation point, and $\mathscr{W}$ is a dimensionless parameter.

In the new variables equation (3.11d) reads, for a polytropic gas with $\gamma = \frac{5}{3}$,

$$\bar{\rho} = \frac{1}{\mathscr{W}^3}\left\{\left[\pm\tfrac{1}{2}\mathscr{W}\left(\bar{v}_x \pm \frac{2}{\mathscr{W}}(\mathscr{W}+1)^{\frac{3}{2}}\right)\right]^{\frac{2}{3}} - 1\right\}^3, \tag{3.12}$$

which gives the simple wave relationship (2.2) with $\phi = v_x$ and the constraint $(\bar{\rho} > 0)$

$$\pm \bar{v}_x \geqslant \frac{2}{\mathscr{W}}[1 - (\mathscr{W}+1)^{\frac{3}{2}}]. \tag{3.13}$$

Let us calculate the breaking time for the progressive simple wave solution (3.12) (+ sign):

$$\frac{\mathrm{d}\lambda_M^{(\pm)}}{\mathrm{d}\xi} = \frac{\mathrm{d}\lambda_M^{(+)}}{\mathrm{d}v_x}\frac{\mathrm{d}v_x}{\mathrm{d}\xi},$$

but according to (2.15)

$$\left.\begin{aligned} \bar{v}_x &= \bar{v}_0 \sin\frac{2\pi}{d}\xi \\ \bar{v}_0 &= v_0/c_{s0} \end{aligned}\right\} \quad (\xi \in [0, d]),$$

and from (3.10) one obtains

$$\frac{\mathrm{d}\lambda_{\mathrm{M}}^{(+)}}{\mathrm{d}\xi} = -\frac{2\pi}{d}\frac{\frac{8}{3}\bar{\rho}^{\frac{2}{3}}+3\mathscr{W}\bar{\rho}}{\bar{\rho}^{\frac{2}{3}}+\mathscr{W}\bar{\rho}}\bar{v}_0 \cos\frac{2\pi}{d}\xi.$$

The breaking time is then

$$\left.\begin{aligned} \frac{2\pi}{d}t_{\mathrm{B}}c_{\mathrm{s}0} &= \inf_{\xi} \psi(\xi), \\ \psi(\xi) &= \left(\frac{\mathrm{d}\lambda_{\mathrm{M}}^{(+)}}{\mathrm{d}\xi}\right)^{-1}. \end{aligned}\right\} \tag{3.14}$$

It can be checked that when $\mathscr{W} = 0$ (no magnetic field) the above formulas reduces to those appropriate for the purely fluid dynamic case.

The infimum of ψ is found numerically and the constraint (3.13) is taken into account.

4. Test fluid for special relativistic magneto-fluid dynamics

The main equations for a relativistic electromagnetically interacting fluid are the energy-momentum conservation, the mass conservation, and the Maxwell equations, together with the Einstein equations which give us the metric.

An easier situation is obtained when we neglect the gravitational field of the fluid in comparison with the background gravitational one (neglecting Einstein's equations): in this case we deal with a 'test fluid'. The equations for such a fluid have been formulated by Lichnerowicz (1967): within a material medium, a general electromagnetic field is represented by two skew-symmetric field tensors $H_{\alpha\beta}$ and $G_{\alpha\beta}$ which satisfy the Maxwell equations

$$\nabla_\alpha(\epsilon^{\alpha\beta\gamma\delta}H_{\gamma\delta}) = 0, \quad \nabla_\alpha G^{\alpha\beta} = J^\beta, \tag{4.1 a, b}$$

where $H_{\alpha\beta}$ is called the electric field–magnetic induction tensor, and $G_{\alpha\beta}$ the magnetic field–electric induction tensor, while $\epsilon^{\alpha\beta\gamma\delta}$ is the Levi–Civita tensor and ∇_α denotes the covariant derivative with respect to the metric.

Generally one can define covariant electric and magnetic fields in the comoving frame by

$$E_\alpha = H_{\alpha\beta}u^\beta, \tag{4.2}$$

$$b_\alpha = \mu h_\alpha = \tfrac{1}{2}\epsilon_{\alpha\beta\gamma\delta}u^\beta H^{\gamma\delta}, \tag{4.3}$$

where u^α is the fluid 4-velocity, and μ is the magnetic permeability assumed constant.

We observe that by the antisymmetry of $H^{\alpha\beta}$,

$$E_\alpha u^\alpha = b_\alpha u^\alpha = 0.$$

The current density in a plasma is given by

$$J^\alpha = \nu u^\alpha + \sigma^{\alpha\beta}E_\beta, \tag{4.4}$$

where ν is the charge density measured by a comoving observer and $\sigma^{\alpha\beta}$ is the conductivity tensor.

The usual assumption of relativistic magneto-fluid dynamics (RMFD) is to assume the medium isotropic ($\sigma^{\alpha\beta} \approx \sigma_0 g^{\alpha\beta}$) with an infinite conductivity ($\sigma_0 \rightarrow \infty$): such a situation may occur for hot and dense plasmas (Bakenstein & Oron 1978).

Then from (4.4), in order to maintain finite the current density, it follows that

$$E_\alpha = 0 \tag{4.5}$$

which, in the classical limit, corresponds to the well known MFD condition

$$\boldsymbol{E}+\frac{\boldsymbol{v}}{c}\times\mu\boldsymbol{H} = 0.$$

In the RMFD approach, $G_{\alpha\beta} = \mu H_{\alpha\beta}$ and J^α does not appear anywhere in the equations: it is defined by the Maxwell equations (4.1*a*).

In the following we shall consider the flow for a test relativistic magneto-fluid in the flat space–time of special relativity: the equations are then

$$\partial_\alpha T^{\alpha\beta} = 0, \quad \partial_\alpha(\rho u^\alpha) = 0, \quad \partial_\alpha(u^\alpha h^\beta - u^\beta h^\alpha) = 0, \tag{4.6a–c}$$

where $T^{\alpha\beta}$ is the total energy-momentum tensor, ρ is the rest mass density, and $\operatorname{diag} g_{\alpha\beta} = (-1, 1, 1, 1)$.

For a perfect fluid (non-dissipative) $T^{\alpha\beta}$ reads (Dixon 1978):

$$T^{\alpha\beta} = (e+p+\mu|h|^2)\, u^\alpha u^\beta + (p+\tfrac{1}{2}\mu|h|^2)\, g^{\alpha\beta} - \mu h^\alpha h^\beta,$$

where p is the pressure, $|h|^2 = h_\alpha h^\alpha > 0$, and e is the total energy density,

$$e = \rho c^2 + \rho\epsilon, \tag{4.7}$$

with ϵ the internal energy, and c the light velocity.

The fluid quantities ρ, p, e (all measured in the local rest frame) are related by the first law of thermodynamics:

$$\theta\,\mathrm{d}s = \mathrm{d}\left(\frac{e}{\rho}\right) + p\,\mathrm{d}\left(\frac{1}{\rho}\right). \tag{4.8}$$

where s is the specific entropy and θ the absolute temperature.

To the system (4.6) we must add an appropriate state equation of the form

$$e = e(p, s) \tag{4.9}$$

For a monatomic relativistic perfect gas, Synge (1957) derived an equation that looks like the usual classical one:

$$\left.\begin{aligned} p &= \rho k_{\mathrm{B}}\theta, \\ e+p &= \rho G(z)\, c^2, \\ z &= mc^2/K_{\mathrm{B}}\theta, \\ G(z) &= K_3(z)/K_2(z), \end{aligned}\right\} \tag{4.10}$$

where k_{B} is the Boltzmann constant and $K_n(z)$ are the modified Bessel functions of the second kind (Magnus & Oberhettinger 1966). For such a gas, the ratio γ of the heat capacities is (De Groot, Van Leeuwen & Van Weert 1980)

$$\gamma = \frac{z^2 \dfrac{\mathrm{d}G}{\mathrm{d}z}}{1+z^2\dfrac{\mathrm{d}G}{\mathrm{d}z}}. \tag{4.11}$$

We observe that γ is a function of the temperature, and it is possible to prove that, in the Newtonian limit $(z \gg 1), \gamma = \frac{5}{3}$, whereas in the ultrarelativistic case $(z \ll 1)$, $\gamma = \frac{4}{3}$.

In the following we shall consider a relativistic perfect gas with constant heat capacities, e.g. a polytropic gas

$$p = K(s)\,\rho^{\gamma}, \tag{4.12}$$

with $\gamma = \frac{5}{3}$ and $\gamma = \frac{4}{3}$ (Newtonian and relativistic cases); in this case the internal energy density is

$$\epsilon = \frac{p}{\gamma - 1}.$$

The sound speed is given by

$$c_s^2 = \frac{\partial p}{\partial e} c^2 = \frac{\gamma p(\gamma-1)\,c^2}{\rho c^2(\gamma-1)+\gamma p} \leqslant c^2(\gamma-1). \tag{4.13}$$

Another interesting state equation is the barotropic one, where

$$e = e(p).$$

To this class, belongs the state equation for a radiation-dominated gas,

$$p = (\gamma-1)\,e, \tag{4.14}$$

with $\gamma = \frac{4}{3}$.

From the hypothesis of one-dimensional motion we obtain from equation (4.6*c*) the invariant of the motion

$$J_0 = u^0 h^1 - u^1 h^0 = \text{const.} \tag{4.15}$$

By writing

$$u^{\alpha} = \Gamma(1, v_x/c, v_y/c, v_z/c), \quad h^{\alpha} = (h^0, h^1, h^2, h^3) \tag{4.16}$$

where Γ is the Lorentz factor, it follows from $u_{\alpha} h^{\alpha} = 0$ that

$$J_0 = \Gamma\{(c^2 - v_x^2)\,h^1 - v_x v_y h^2 - v_x v_z h^3\}/c^2$$

which, in the non-relativistic limit, coincides with H_x.

Manipulating system (4.6) it is possible to see that such equations are equivalent to (Anile & Pennisi 1987)

$$u^{\alpha}\partial_{\alpha} e + (e+p)\,\partial_{\alpha} u^{\alpha} = 0, \tag{4.17}$$

$$u^{\alpha}\partial_{\alpha} s = 0, \tag{4.18}$$

$$u^{\alpha}\partial_{\alpha} h^{\beta} - h^{\alpha}\partial_{\alpha} u^{\beta} + \frac{1}{e+p}(u^{\beta}h^{\alpha} - h^{\beta}u^{\alpha}e'_p)\,\partial_{\alpha} p = 0, \tag{4.19}$$

$$\begin{aligned}(e+p+\mu|h|^2)\,u^{\alpha}\partial_{\alpha} u^{\sigma} - \mu h^{\alpha}\partial_{\alpha} h^{\sigma} + (h^{\sigma\alpha} + u^{\sigma}u^{\alpha})\,\mu h_{\nu}\partial_{\alpha} h^{\nu} \\ + \frac{1}{e+p}\{(e+p)\,h^{\sigma\alpha} - e'_p\mu|h|^2 u^{\sigma}u^{\alpha} + \mu h^{\sigma}h^{\alpha}\}\,\partial_{\alpha} p = 0,\end{aligned} \tag{4.20}$$

where $e'_p = (\partial e/\partial p)_s = c^2/c_s^2$.

Equation (4.17) is the energy conservation one, (4.18) is the adiabaticity condition, (4.19) are the Maxwell equations and (4.20) the Euler equation. This system can now be written in the form (2.1), with

$$\boldsymbol{u} = (u^{\alpha}, h^{\alpha}, p, s)^{\mathrm{T}},$$

$$\boldsymbol{A}^{\alpha} = \begin{vmatrix} Eu^{\alpha}\,\delta^{\tau}_{\nu}, & -\mu h^{\alpha}\,\delta^{\tau}_{\nu} + p^{\tau\alpha}\mu h_{\nu}, & m^{\nu\alpha}, & 0^{\tau\alpha} \\ h^{\alpha}\,\delta^{\tau}_{\nu}, & -u^{\alpha}\,\delta^{\tau}_{\nu}, & f^{\alpha\tau}, & 0^{\alpha\tau} \\ \eta\,\delta^{\alpha}_{\nu}, & 0^{\alpha}_{\nu}, & e'_p\,u^{\alpha}, & 0^{\alpha} \\ 0^{\alpha}_{\nu}, & 0^{\alpha}_{\nu}, & 0^{\alpha}, & u^{\alpha} \end{vmatrix}, \tag{4.21}$$

where $0^{\tau\alpha}, 0^{\alpha}$, indicate tensors and vectors with vanishing components, and $\eta = e+p$, $E = \eta+\mu|h|^2$, $p^{\tau\alpha} = h^{\tau\alpha}+u^{\tau}u^{\alpha}$, $h^{\alpha\beta} = g^{\alpha\beta}+u^{\alpha}u^{\beta}$, $m^{\tau\alpha} = (\eta h^{\tau\alpha}-e'_p\mu|h|^2 u^{\tau}u^{\alpha}+\mu h^{\tau}h^{\alpha})/\eta$, $f^{\tau\alpha} = (u^{\tau}h^{\alpha}e'_p-u^{\alpha}h^{\tau})/\eta$.

A detailed study of the mathematical structure of system (2.21) (eigenvalues, eigenvectors, hyperbolicity) has been performed by Anile & Pennisi (1987).

In particular, in the case of one-dimensional flow, the determinant of $\boldsymbol{A}_n = \boldsymbol{A}^1$ is

$$\det \boldsymbol{A}_n = Ea^2A^2N_4, \tag{4.22}$$

where

$$\left.\begin{aligned} a = \Gamma(v_x-\lambda)/c, \quad B = h^1-\lambda h^0/c, \quad G = 1-\lambda^2/c^2, \quad A = Ea^2-B^2, \\ N_4 = \eta(e'_p-1)\,a^4-(\eta+e'_p\mu|H|^2)\,a^2G+B^2G. \end{aligned}\right\} \tag{4.23}$$

The solutions corresponding to $a = 0, A = 0, N_4 = 0$ represent material, Alfvén and magnetoacoustic waves respectively.

Let us consider some particular cases.

4.1. *Relativistic fluid dynamics case*

For $h^{\alpha} \equiv 0$ we reduce to the fluid dynamic case: for purely longitudinal motion $(v_y = v_z = 0)$, $\boldsymbol{u} = (s,p,v)^{\mathrm{T}}$ with $v = v_x$ and the eigenvalues of $\boldsymbol{A}_n$ are

$$\lambda^{(0)} = v \text{ (material waves)}, \quad \lambda^{(\pm)} = \frac{v\pm c_s}{1\pm vc_s/c^2} \text{ (acoustic waves)}. \tag{4.24a, b}$$

The Riemann invariants associated with acoustic waves are, besides J_0 (Liang 1977)

$$J_1 = s, \quad J_{\pm} = \frac{1}{2c}\log\frac{c+v}{c-v}\mp\int\frac{\mathrm{d}p}{(e+p)\,c_s}. \tag{4.25a, b}$$

We shall consider two kinds of state equation, from which we can express $c_s = c_s(v)$, and consequently $\boldsymbol{u} = \boldsymbol{u}(\phi)$ with $\phi = v$.

4.1.1. *Relativistic polytropic gas: $p = k\rho^{\gamma}$*

From (4.25b) we obtain

$$c_s = (\gamma-1)^{\frac{1}{2}}\frac{\left(\dfrac{c+v}{c-v}\right)^{\mp\frac{1}{2}(\gamma-1)^{\frac{1}{2}}}+\dfrac{c_{s0}+c(\gamma-1)^{\frac{1}{2}}}{c_{s0}-c(\gamma-1)^{\frac{1}{2}}}}{\dfrac{c_{s0}+c(\gamma-1)^{\frac{1}{2}}}{c_{s0}-c(\gamma-1)^{\frac{1}{2}}}-\left(\dfrac{c+v}{c-v}\right)^{\mp\frac{1}{2}(\gamma-1)^{\frac{1}{2}}}}\,c, \tag{4.26}$$

with c_{s0} the sound speed at $v = 0$. From the constraints $0 \leqslant c_s \leqslant c(\gamma-1)^{\frac{1}{2}}$ we obtain for progressive waves (minus sign)

$$-1 < \frac{h-1}{h+1} \leqslant \frac{v}{c} < 1, \quad h = \left(\frac{c_{s0}+c(\gamma-1)^{\frac{1}{2}}}{c(\gamma-1)^{\frac{1}{2}}-c_{s0}}\right)^{-2(\gamma-1)^{-\frac{1}{2}}}, \tag{4.27}$$

and for regressive ones (plus sign)

$$-1 \leqslant \frac{v}{c} \leqslant \frac{h-1}{h+1} < 1, \quad h = \left(\frac{c_{s0}+c(\gamma-1)^{\frac{1}{2}}}{c(\gamma-1)^{\frac{1}{2}}-c_{s0}}\right)^{2(\gamma-1)^{-\frac{1}{2}}}. \tag{4.28}$$

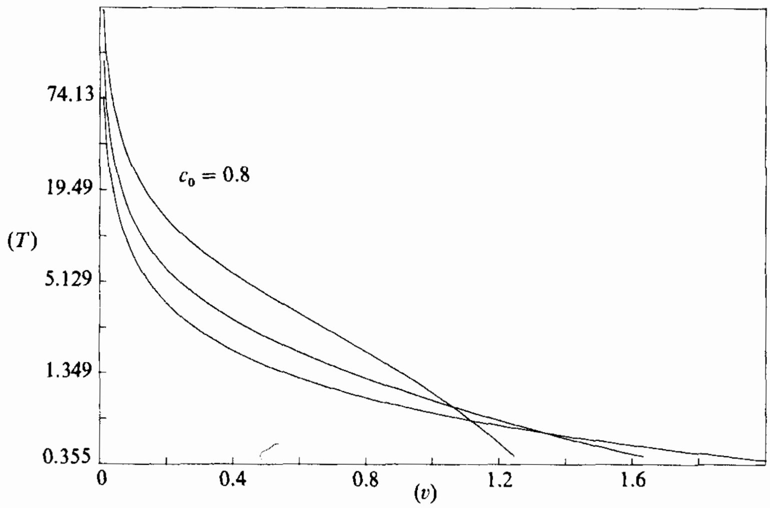

FIGURE 1. Relativistic and classic fluid dynamics $\gamma = \frac{5}{3}$: $(T) = 2\pi t_B c_{s0}/d$ versus $(v) = v_0/c_{s0}$. The lower curve corresponds to the fluid case; the middle one is obtained for $c_0 = 0.4$.

If we introduce the non-dimensional variables

$$\bar{v} = \frac{v}{c_{s0}}, \quad \bar{c} = \frac{c_{s0}}{c},$$

the breaking time, according to the definition, is

$$\frac{2\pi t_B c_{s0}}{d} = \inf_{\xi} \psi(\xi),$$

$$\left.\begin{aligned} \psi(\xi) &= -\frac{1}{\bar{v}_0 \cos\frac{2\pi}{d}\xi} \frac{[K - \tau \pm \bar{v}_0 \bar{c} \sin\frac{2\pi\xi}{d}(\tau + K)(\gamma-1)^{\frac{1}{2}}]^2}{(2-\gamma)K^2 + (2-\gamma)\tau^2 + 2\tau K(1-2\gamma)}, \\ \tau &= \left[\frac{1+\bar{v}\bar{c}}{1-\bar{v}\bar{c}}\right]^{\mp\frac{1}{2}(\gamma-1)}, \quad k = \frac{\bar{c} + (\gamma-1)^{\frac{1}{2}}}{\bar{c} - (\gamma-1)^{\frac{1}{2}}}, \quad \bar{v} = \bar{v}_0 \sin\frac{2\pi}{d}\xi, \end{aligned}\right\} \tag{4.29}$$

where the first sign is associated with $\lambda^{(+)}$.

If $\bar{c} \ll 1$, the above formulae reduce to those of the non-relativistic case treated in §3.1.

The infimum of (4.29) is found numerically, taking into account the constraints (4.27), (4.28), i.e. the computation stops if those inequalities do not hold. In figure 1 we compare the case with $\gamma = \frac{5}{3}$ (both progressive and regressive) with the non-relativistic case (3.9) for several values of the parameter $\bar{c}$.

4.1.2. *Barotropic gas: $p = (\gamma - 1)e, c_s^2 = (\gamma-1)c^2$*

From (4.24*a*) we evaluate $d\lambda/d\xi$ and, after lengthy calculations, find

$$t_B = \frac{d}{4\pi c} \frac{(\gamma-1)^{\frac{1}{2}}}{\left[2(1+8v_0^2(\gamma-1)/c^2)^{\frac{1}{2}} - 4\frac{v_0^2}{c^2}(\gamma-1) - 2\right]^{\frac{1}{2}}} \frac{[3-(1+8v_0^2(\gamma-1)/c^2)^{\frac{1}{2}}]^2}{2-\gamma}. \tag{4.30}$$

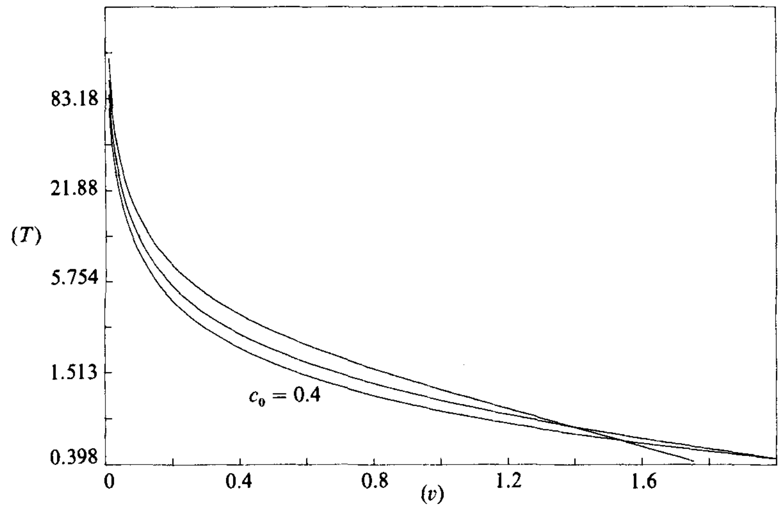

FIGURE 2. Relativistic fluid dynamic case with $\gamma = \frac{4}{3}$: $(T) = 2\pi t_B c_{s0}/d$ versus $(v) = v_0/c_{s0}$. The upper curve corresponds to the barotropic case; the middle one is obtained for $c_0 = 0.8$.

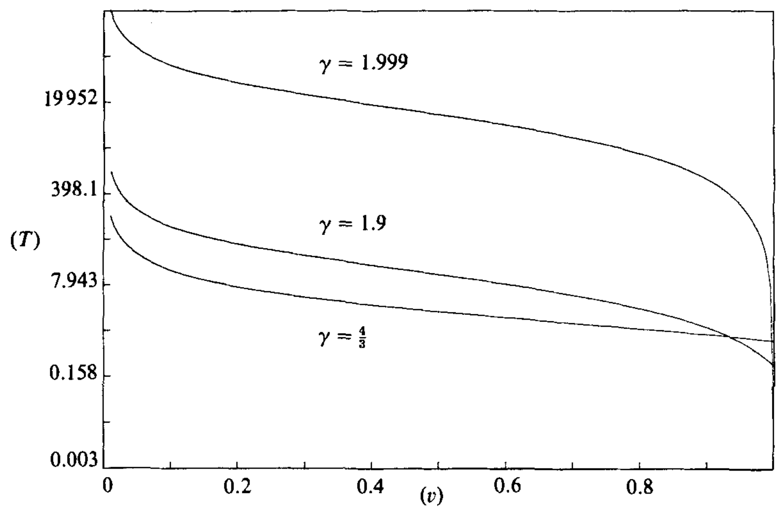

FIGURE 3. Barotropic relativistic fluid dynamics: $(T) = 2\pi t_B c_{s0}/d$ versus $(v) = v_0/c_{s0}$ for several values of γ.

For $\gamma = 2$ we have an incompressible fluid, and if $v_0 \neq c$ then $t_B \to +\infty$: this is an exceptional case, in which the shock is never formed. If $v_0 = c$ and $\gamma \to 2$ then $t_B \to 0$: this behaviour is plotted in figure 3.

4.2. *Magneto-fluid-dynamic case with $v_y = v_z = 0$*

Simple wave solutions associated with (4.21) have been found explicitly, in some cases (Anile & Muscato 1988); that happens when the fluid's motion is purely longitudinal and the magnetic field at a stagnation point is purely transverse,

$$v_y = v_z = 0, \quad h^1(v=0) = 0. \tag{4.31}$$

Hence from the invariant J_0 it follows, throughout the flow, that

$$h^1 = 0. \tag{4.32}$$

Under the above hypothesis let us evaluate h^α from the definition (4.2), using the appropriate electromagnetic tensor (Landau & Lifshitz 1959):

$$h^0 = 0, \quad h^1 = \Gamma H_x = 0, \quad h^2 = H_y, \quad h^3 = H_z,$$

where H_x, H_y, H_z are the components of the magnetic field in the orthogonal Eulerian frame of reference. The field vector reduces to

$$\boldsymbol{u} = (\Gamma, v, H_y, H_z, p, s)^{\mathrm{T}}.$$

This case is the analogue of the previous one treated in §3.

Because $B = h^1 - \lambda h^0 = 0$ from (4.23), the Alfvén and the slow MFD waves coincide with the material one; the fast one is given by (we put $v = v_x$)

$$\left.\begin{aligned} \lambda_{\mathrm{f}}^{(\pm)} &= \frac{v \pm \bar{\lambda}}{1 \pm v\bar{\lambda}/c^2}, \\ \bar{\lambda} &= \left(\frac{(e+p)\,c_{\mathrm{s}}^2/c^2 + \mu|H|^2}{e+p+\mu|H|^2}\right)^{\frac{1}{2}} c, \end{aligned}\right\} \tag{4.33}$$

the associated Riemann invariants are

$$J_1 = s, \quad J_2 = \frac{H_y}{\exp\displaystyle\int \frac{\mathrm{d}p}{(e+p)\,c_{\mathrm{s}}^2/c^2}}, \tag{4.34a,b}$$

$$J_3 = \frac{H_z}{\exp\displaystyle\int \frac{\mathrm{d}p}{(e+p)\,c_{\mathrm{s}}^2/c^2}}, \quad J_\pm = \frac{1}{2c}\log\frac{c+v}{c-v} \mp \int \frac{\bar{\lambda}}{(e+p)\,c_{\mathrm{s}}^2}\,\mathrm{d}p, \tag{4.34c,d}$$

with the sign $\mp$ associated with $\lambda_{\mathrm{f}}^{(\pm)}$.

The invariants J_2, J_3 give us explicitly H_y, H_z as functions of p; $J_\pm$ gives us a relationship between v and p that in general must be solved numerically. We write this relation in the form of a differential equation

$$\frac{\mathrm{d}p}{\mathrm{d}v} = \pm\frac{(e+p)\,\Gamma^2 c_{\mathrm{s}}^2}{c^2\bar{\lambda}}, \tag{4.35}$$

which enables us to express $\boldsymbol{u} = \boldsymbol{u}(\phi)$ with $\phi = v$.

As before we shall associate with (4.34) the barotropic state equation and the polytropic one.

4.2.1. *Relativistic barotropic gas*: $p = (\gamma - 1)\,e, c_{\mathrm{s}}^2 = (\gamma-1)\,c^2$

From (4.34a, b) we obtain

$$\left.\begin{aligned} H_y &= J_z p^{1/\gamma}, \\ H_z &= J_3 p^{1/\gamma}. \end{aligned}\right\} \tag{4.36}$$

Let p_0 be a reference pressure; then we can introduce the non-dimensional variables

$$\bar{p} = \frac{p}{p_0}, \quad \bar{v} = \frac{v}{c_{\mathrm{s}}} = \frac{v}{c(\gamma-1)^{\frac{1}{2}}}, \tag{4.37}$$

and because $-1 \leqslant v/c \leqslant 1$, we have the constraint $-1 \leqslant \bar{v}(\gamma-1)^{\frac{1}{2}} \leqslant 1$.

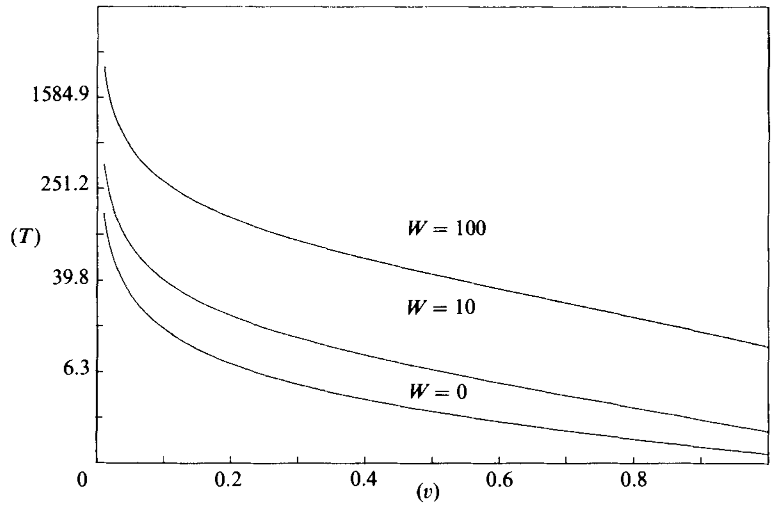

FIGURE 4. Barotropic RMFD with $\gamma = \frac{4}{3}$: $(T) = 2\pi t_{\mathrm{B}} c_{\mathrm{s0}}/d$ versus $(v) = v_0/c_{\mathrm{s0}}$: the lower curve corresponds to the fluid dynamic case.

The breaking time for $\gamma = \frac{4}{3}$, according to the definition, is given by

$$\left.\begin{aligned} \frac{2\pi t_{\mathrm{B}} c_{\mathrm{s}}}{d} &= \inf_{\xi} \psi(\xi), \\ \psi(\xi) &= -\frac{9}{32} \frac{(4\bar{p}+W\bar{p}^{\frac{3}{2}})(4\bar{p}/3+W\bar{p}^{\frac{3}{2}})(1 \pm \bar{\lambda}\bar{v})^2}{\bar{p}(\bar{p}+W\bar{p}^{\frac{3}{2}})\,\bar{v}_0 \cos\dfrac{2\pi}{d}\xi}, \\ \bar{\lambda} &= \left(\frac{4\bar{p}/3+W\bar{p}^{\frac{3}{2}}}{4\bar{p}+W\bar{p}^{\frac{3}{2}}}\right)^{\frac{1}{2}}, \quad \bar{v} = \bar{v}_0 \sin\frac{2\pi}{d}\xi, \end{aligned}\right\} \tag{4.38}$$

where $W = \mu|H|_0^2/p_0 = \mu(J_2^2+J_3^2)\,p_0^{\frac{1}{2}}$, a non-dimensional parameter, and the sign $\pm$ is associated with $\lambda_{\mathrm{f}}^{(\pm)}$.

The fluid case (without magnetic field) is obtained for $W = 0$. The infimum of (4.38) is found numerically by solving the differential equation (4.37): the initial conditions are chosen such that

$$p(v = 0) = p_0.$$

In figure 4 we compare the breaking times for barotropic fluid ($\gamma = \frac{4}{3}$) and magneto-fluid. We observe that the effect of the magnetic field is to raise the curves with respect to the purely fluid case.

4.2.2. *Relativistic polytropic gas*: $p = K\rho^{\gamma}$

From (4.34*b*, *c*) we obtain the same formulae (4.36), and (4.35) enables us to express $p = p(v)$.

Let p_0 and c_{s0} be respectively the pressure and the sound velocity at the stagnation point; then we can introduce

$$\bar{v} = \frac{v}{c_{\mathrm{s0}}}, \quad \bar{p} = \frac{p}{p_0}.$$

The breaking time is

$$\left.\begin{aligned} \frac{2\pi t_B c_{s0}}{d} &= \inf_{\xi} \psi(\xi), \\ \psi(\xi) &\frac{-(1\pm\bar{\lambda}\bar{c}\bar{v})^2}{1-\bar{\lambda}^2 \pm \dfrac{d\bar{\lambda}}{d\bar{v}}\dfrac{(1-\bar{v}^2\bar{c}^2)}{\bar{c}}} \frac{1}{\bar{v}_0 \cos\left(\dfrac{2\pi}{d}\xi\right)}, \\ \bar{v} = \bar{v}_0 \sin\frac{2\pi}{d}\xi, \quad & \bar{\lambda}^2 = \frac{\gamma\bar{p}+W\bar{p}^{2/\gamma}}{\bar{\eta}+W\bar{p}^{2/\gamma}}, \quad \bar{\eta} = L\bar{p}^{1/\gamma} + \frac{\gamma}{\gamma-1}\bar{p}, \end{aligned}\right\} \tag{4.39}$$

with $W = \mu|H|_0^2/p_0$, $\bar{c} = c_{s0}/c$, $L = \rho_0 c^2/p_0$, where the sign $\pm$ is associated with $\lambda_f^{(\pm)}$.

In order to find the infimum of (4.39) we must solve (4.37) by a numerical integration: the initial condition is

$$p(v=0) = p_0$$

that is analogous to the relativistic-fluid-dynamic case where $c_s(v=0) = c_{s0}$; now the thermodynamical constraint is $0 \leqslant c_s^2 \leqslant c^2(\gamma-1)$ which is imposed during the integration.

The infimum of (4.39) is found numerically for $\gamma = \frac{5}{3}$, and taking account of the two parameters W (related to the strength of magnetic field, and $\bar{c}$ (related to the Riemann invariant $J\pm$).

In the non-relativistic MFD regime we introduced the parameter

$$\mathscr{W} = \frac{\mu H_0^2/\rho_0}{c_{s0}^2};$$

the relationship between W and $\mathscr{W}$ is

$$W = \mathscr{W}\bar{c}^2 \frac{\gamma(\gamma-1)-\gamma\bar{c}^2}{\bar{c}^2(\gamma-1)}. \tag{4.40}$$

In figure 5 we fix $\mathscr{W} = 1$, allowing $\bar{c}$ to vary, with $\gamma = \frac{5}{3}$: if $\bar{c} \to 0$ our curves approach the Newtonian MFD case.

The above formulae reduce to the corresponding non-relativistic cases (with or without magnetic field) for $\bar{c} \ll 1$.

5. Conclusions

We employed two algorithms in order to find the infimum of $\psi(\xi)$: the first one operates by dividing the interval allowed to ξ and evaluating the function at these points.

With the second one we looked for the zeros of $d\psi/d\xi$ is a suitable interval, and then the infimum is found between these zeros.

With such methods t_B is valued with a precision of five significant digits.

In figure 1 we compare, in logarithm scale, the non-relativistic fluid case (lower curve) with the relativistic one for $\gamma = \frac{5}{3}$ (the middle curve is obtained for $\bar{c} = 0.4$, the upper for $\bar{c} = 0.8$).

The relativistic curves stop because we have the constraints (4.27), (4.28); the

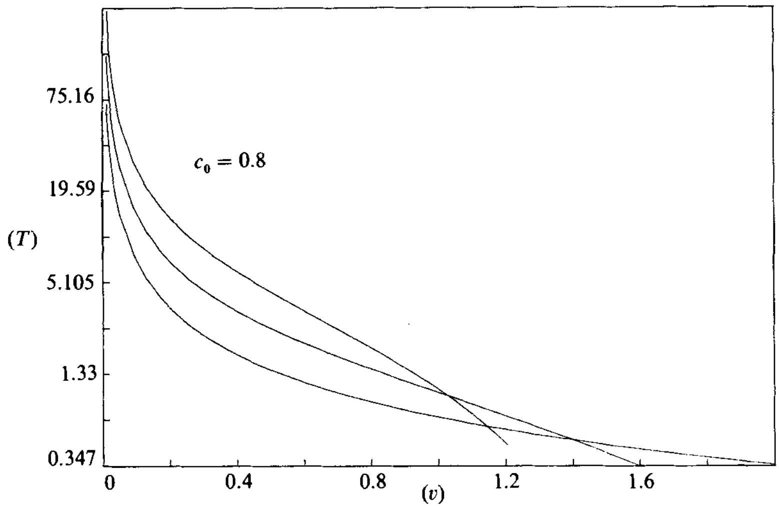

FIGURE 5. Polytropic RMFD and classical MFD with $\gamma = \frac{5}{3}$, $\mathscr{W} = 1$: $(T) = 2\pi t_B c_{s0}/d$ versus $(v) = v_0/c_{s0}$. The lower curve corresponds to the classic case; the middle one is obtained for $c_0 = 0.6$.

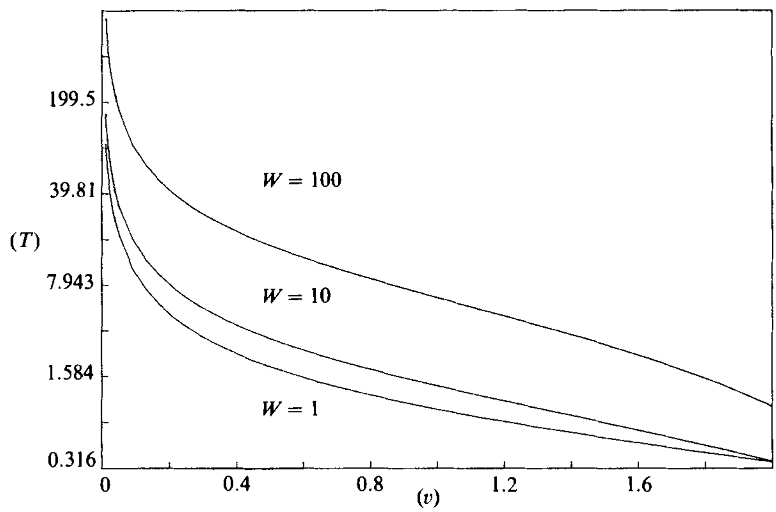

FIGURE 6. Polytropic RMFD and classical MFD with $\gamma = \frac{5}{3}, \bar{c} = 0.4$: $(T) = 2\pi t_B c_{s0}/d$ versus $(v) = v_0/c_{s0}$ for several values of W.

effect of relativity is to increase the breaking time with respect to the Newtonian case, but when $\bar{v}_0$ approaches the limit value, the breaking time decreases quickly.

In figure 2 we compare the relativistic fluid dynamic cases with $\gamma = \frac{4}{3}$: the upper curve represents the barotropic case (4.30), the other ones the polytropic case (4.29) for two values of the parameter $\bar{c}$.

The barotropic cases (which have no Newtonian analogues) are compared in figure 4 for $\gamma = \frac{4}{3}$: the effect of the magnetic field is to increase the breaking time, i.e. the magnetic pressure slows down the breakdown.

In figure 5 we compare the polytropic RMFD case (4.39) with the classical one (3.14): we fix $\gamma = \frac{5}{3}$ and $\mathscr{W} = 1$, varying $\bar{c}$. Vice versa in figure 6, we fix $\bar{c} = 0.4$ and vary W: again the effect of the magnetic pressure is to increase the breaking time when W increases.

From the graphs we notice that in all regimes (classical and relativistic) t_B is a decreasing function of v_0 and an increasing function of the wavelength; moreover we obtained the same results for progressive and regressive waves.

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REFERENCES

AMSDEN, A. A., GOLDHABER, A. S., HARLOW, F. H. & NIX, J. R. 1978 *Phys. Rev.* C **17**, 2080.

ANILE, A. M. & MUSCATO, O. 1988 Magnetofluiddynamic simple waves in special relativity. *Ann. Inst. H. Poincaré* **48**, 1.

ANILE, A. M. & PENNISI, S. 1987 *Ann. Inst. H. Poincaré* **46**, 27.

ANILE, A. M. & RUSSO, G. 1986 *Phys. Fluids* **29**, 2847.

BEGELMAN, M. C., BLANDFORD, R. D. & REES, M. J. 1984 *Rev. Mod. Phys.* **56**, **1**, 255.

BEKENSTEIN, J. D. & ORAN, E. 1978 *Phys. Rev.* **D 18**, 1809.

BOILLAT, G. 1970 *J. Math. Phys.* **11**, 1484.

CABANNES, H. 1970 *Theoretical Magnetofluiddynamics*. Academic.

CLARE, R. B. & STROTTMAN, D. 1986 *Phys. Rep.* **141**, 177.

DE GROOT, S. R., VAN LEEUWEN, W. A. & VAN WEERT, CH. G. 1980 *Relativistic Kinetic Theory*. North Holland.

DIXON, W. G. 1978 *Special Relativity*. Cambridge University Press.

FERRARI, A., TRUSSONI, E. & ZANINETTI, L. 1983 *Astrophys. J.* **125**, 179.

HAWLEY, J. F., SMARR, L. L. & WILSON, J. R. 1984 *Astrophys. J. Suppl.* **55**, 221.

JEFFREY, A. 1976 *Quasilinear Hyperbolic Systems and Waves*. Pitman.

LANDAU, L. D. & LIFSHITZ, E. M. 1959 *Fluid Mechanics*. Pergamon.

LIANG, E. P. T. 1977 *Astrophys. J.* **211**, 361.

LIBERMANN, M. A. & VELIKONICH, A. L. 1986 *Physics of Shock Waves in Gases and Plasmas*. Springer.

LICHNEROWICZ, A. 1967 *Relativistic Hydrodynamic and Magnetohydrodynamics*. Benjamin.

MAJDA, A. & ROSALES, R. 1984 *Stud. Appl. Maths* **71**, 149.

MAGNUS, W. & OBERHETTINGER, F. 1966 *Formulas and Theorems for Special Functions of Mathematical Physics*. Springer.

MILLER, R. B. 1982 *Intense Charged Particle Beans*. Plenum.

TAUSSIG, R. 1973 Shock wave production of relativistic plasmas. In *Dynamics of Ionized Gas* (ed. M. J. Lightill, I. Imai & H-Sati). University of Tokyo Press.

SYNGE, J. L. 1957 *The Relativistic Gas*. North Holland.

THOMPSON, K. W. 1986 *J. Fluid Mech.* **171**, 365.

WHITHAM, G. 1974 *Linear and Nonlinear Waves*. Interscience.